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A Simple Approach to Cardinal Lagrange and Periodic Lagrange Splines

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We derive via a simple formula an explicit expression for the cardinal Lagrange spline as a combination of decreasing null splines and a polynomial correction term near the origin. Many qualitative properties, new and old, are obtained as immediate consequences. The periodic analogue is also discussed. The method is much simpler than those in the previous work of Nilson, ter Morsche, Reimer, and the author, and is readily generalized to cardinal L-splines. © 1986 Academic Press, Inc.

1. INTRODUCTION

We consider the cardinal spline interpolation problem of finding an element in $\mathbb{S}_n$ which interpolates to some given function f at the integers Z, where $\mathbb{S}_n$ is the space of cardinal splines of degree n having (simple) knots at $-h+Z$, with $0 \leqslant h < 1$ fixed. We assume that the data $f(j)$ are of power growth and require the spline interpolant to be of power growth also. Furthermore we always assume $h \neq (1-(-)^n)/4$ so that existence and uniqueness of the interpolant is guaranteed. (See Schoenberg [11, 13] and Micchelli [4, 5] for details. See also Meinardus and Merz [3].)

The interpolant will be denoted by $S_n f$, or Sf as long as n is fixed, and is given in Lagrange form by

$$Sf(x) = \sum_{j=-\infty}^{\infty} f(j)\, L(x-j), \tag{1}$$

where the cardinal Lagrange spline $L := L_{n,Z-h}$ is the unique element in $\mathbb{S}_n$ that interpolates, at the integers, the unit data sequence $f(j) = \delta_{0,j}$, $j \in Z$.

If the data f are N-periodic, i.e., $f(j) = f(j+N)$, $j \in Z$, with N a positive integer, then so is the interpolant, i.e., $Sf(x+N) = Sf(x)$, all x real. In this case Sf takes the form

$$Sf(x) = \sum_{j=0}^{N-1} f(j)\, L^{(N)}(x-j), \text{ in particular, } L^{(N)}(x) = \sum_{j=-\infty}^{\infty} L(x-jN), \tag{1'}$$

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where $L^{(N)}$is the periodic interpolant to the N-periodic unit data $f(j) = \sum \delta_{0,j+kN}$, i.e., $f(j) = 1$ or 0, depending on whether $j \in NZ$ or not.

In addition to the summetry

$$L(-x) = L(x) \qquad \text{and} \qquad L^{(N)}(-x) = L^{(N)}(x), \text{ with } \quad h = (1+(-)^n)/4,$$

or more generally,

$$L_{n,Z-h}(-x) = L_{n,Z-(1-h)}(x), \qquad \text{with} \quad (1-(-)^n)/4 \neq h \in [0, 1),$$

which is a consequence of the uniqueness of the interpolant, we summarize several properties concerning L and $L^{(N)}$ for ready reference.

(i) Sign regularity of L, i.e.,

$$\operatorname{sgn} L(x) = (-)^j, \qquad 0 < x \in (j, j+1), \tag{2}$$

is proved by de Boor and Schoenberg [2] (see also Micchelli [4]) via the Budan–Fourier theorem and Gantmacher and Krein's theory on oscillation matrices.

(ii) Sign regularity of $L^{(N)}$, i.e.,

$$\operatorname{sgn} L^{(N)}(x) = (-)^j, \qquad 0 \leqslant j < x < j+1, \quad j+1 \leqslant \frac{N-1}{2}, \tag{2'}$$

is proved by Richards [8] via the cyclic variation diminishing property.

(iii) $L(\cdot - j)$ and $L^{(N)}(\cdot - j)$ perform as an "extremal" spline basis (see Reimer [9]), i.e.,

$$\| L \|_\infty = 1 \tag{3}$$

$$\| L^{(N)} \|_\infty = 1. \tag{3'}$$

Even more is true.

$$1 \geqslant L(x) \geqslant -L(x+1) \cdots \geqslant (-)^j L(x+j) \geqslant 0, \qquad x \in [0, 1) \text{ and } j \geqslant 0, \tag{4}$$

and

$$\begin{gathered} 1 \geqslant L^{(N)}(x) \geqslant -L^{(N)}(x+1) \geqslant \cdots \geqslant (-)^j L^{(N)}(x+j) \geqslant 0, \\ x \in [0, 1) \text{ and } 0 \leqslant j \leqslant (N-1)/2, \end{gathered} \tag{4'}$$

are implied in ter Morsche [6], for all n and $h = (1+(-)^n)/4$, and are proved by Reimer for odd n and $h = 0$. Actually, strict inequality holds in (4) and (4′) with nonintegral x. (See Section 3 below.)

(iv) (3) and (3′) are also consequences of the fact that

$$s(x) := \sum_{j=-\infty}^{\infty} (L(x-j))^2 \leqslant 1, \qquad x \quad \text{real.} \tag{5}$$

and

$$s^{(N)}(x) := \sum_{j=0}^{N-1} (L^{(N)}(x-j))^2 \leqslant 1, \qquad x \quad \text{real.} \tag{5′}$$

Siepmann and Sündermann [14] proved (5) and (5′) for the cubic case, $n=3$ and $h=0$, and conjectured these to be true for all odd degree n and $h=0$. It turns out these are true for all degree and $h=(1+(-)^n)/4$, as is proved by the author, with a detailed description of extreme points and monotinicity in [1].

We give below a simple, unified approach to these results, (i) to (iv), except (4′) for even N. In addition, radius of convergence of (1) and the limit of the spline interpolant as the degree tends to infinity are obtained. This gives a quick proof to the results in [13] for l_2 data. The argument uses only a formula (see (6) below) of Micchelli and the interlacing property of the zeros of the generalized Euler–Frobenius polynomials (see [6, 4, 11]). We also obtain new insight into (5) and (5′) (see Eqs. (13) and (13′) below). We note that some expressions, similar to (15) or (15′) below, are obtained in Nilson [7; pp. 448, 450] for odd n and $h=0$, in ter Morsche [6; pp. 216, 217] for $h=(1+(-)^n)/4$, and in Reimer [9; p. 95] for odd n and $h=0$.

Our argument is extremely simple and is readily generalized to cardinal L-splines [5].

All equalities stated below are for general h, all inequalities are for the case $h=(1-(-)^n)/4$, since that $\max |E(x, e^{iu})| \leqslant 1$ (see (10)).

2. $L(x)$ as an Average

Following Schoenberg, we begin with Euler's generating function

$$\left(\frac{t-1}{t-e^z}\right) e^{xz} = \sum_{n=-\infty}^{\infty} A_n(x, t)\frac{z^n}{n!},$$

which defines $A_n(x, t)$ (note that $(t, z)=(0, 0)$ is a removable singularity) and gives $A_n(x, 0)=(x-1)^n$. It is known that $A_n(h, \lambda)=0$ has $m := n-\delta_{0,h}$ roots, all simple and negative,

$$\lambda_m(h) < \lambda_{m-1}(h) < \cdots < \lambda_1(h) \qquad (<0),$$

none of which is equal to -1 (recall that the case $h=(1-(-)^n)/4$ is excluded). We note that, with Q_{n+1}, Schoenberg's forward B-spline of degree n,

$$P_n(x, t) := (t-1)^n A_n(x, t) = n! \sum_{j=0}^{n} t^j Q_{n+1}(j+1-x)$$

is a polynomial in x and t, and is monic in t, i.e.,

$$\Pi_{n,h}(\lambda) := P_n(h, \lambda) = (\lambda-\lambda_1) \cdots (\lambda-\lambda_m), \qquad \text{with} \quad \lambda_j = \lambda_j(h),\ j=1, 2, \ldots, m.$$

Our argument starts with an integral representation of the Lagrange spline [4]

$$L(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} E(x, e^{iu})\, du = \frac{1}{2\pi i} \int_T E(x, z) \frac{dz}{z}, \tag{6}$$

where T is the unit circle, positively oriented, and $E(x, t) := Sf_t(x)$ is the spline interpolant to the complex exponential function $f_t(x) := t^x$, $t \in C$; i.e., $E(\cdot\, t)$ in $\mathbb{S}_n$, which satisfies

$$E(j, t) = t^j, \qquad \text{and, as a consequence,} \qquad E(x+1, t) = tE(x, t). \tag{7}$$

These descriptions of $E(x, t)$ are valid if $t \in C - \{0, 1\}$ is not among the m $\lambda_j(h)$. Actually we have, with such t,

$$E(x, t) = \frac{A_n(x+h, t)}{A_n(h, t)} = \frac{\Pi_{n,x+h}(t)}{\Pi_{n,h}(t)}, \qquad -h \leqslant x \leqslant 1-h. \tag{8}$$

We mention the periodic analogue of (6),

$$L^{(N)}(x-j) = \frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi ik/Nj} E(x, e^{2\pi ik/N}), \tag{6$'$}$$

whose relation to (6) is to be discussed later. We note that (6) and (6′) are immediately justified by (7) and (8).

A first consequence of (6) and (6′) is

$$\| L \|_\infty \leqslant \max | E(x, e^{iu}) | \qquad \text{and} \qquad \| L^{(N)} \|_\infty \leqslant \max | E(x, e^{2\pi ik/N}) |, \tag{9}$$

which, together with the fact that (see [1, 5])

$$\| E(\cdot, e^{iu}) \|_\infty \leqslant 1, \qquad u \text{ real}, \tag{10}$$

implies (3) and (3'). The sharp results

$$-1 < L(x) \leqslant 1, \qquad \text{with equality if and only if } x = 0 \tag{11}$$

and

$$-1 < L^{(N)}(x) \leqslant 1, \qquad \text{with equality if and only if } x \in NZ, \tag{11'}$$

are obtained by the author via a different approach with the aid of [1].

Next we observe that (6) and (8) yield, for each fixed x,

$$L(x-j) = E(x, e^{i\cdot})\hat{}\,(j) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iju} E(x, e^{iu})\, du \tag{12}$$

and

$$s(x) = \| E(x, e^{i\cdot}) \|_{L_2(T)}^2 \leqslant \| E(x, e^{i\cdot}) \|_\infty^2 \leqslant 1, \tag{13}$$

from which (5) follows. We note that (12), in an equivalent form, and therefore (4) implicitly, are obtained in ter Morsche [6] via the aid of a difference operation.

We have also

$$L^{(N)}(x-j) = \sum_{k=-\infty}^{\infty} E(x, e^{i\cdot})\hat{}\,(j+kN) \tag{12'}$$

and

$$s^{(N)}(x) = \frac{1}{N} \sum_{k=0}^{N-1} | E(x, e^{2\pi ik/N}) |^2 \tag{13'}$$

from which (5') follows.

Let us mention that (13') follows from (12') and the following general fact:

If the continuous function $f(u)$ is 2π-periodic and the Fourier series $\sum f\hat{}\,(j)\, e^{iju}$ converges to f at $u = 2\pi k/N$, $k = 0, 1, \dots, N-1$, then we obtain, as can be proved by an exchange of the order in a double sum, that

$$\sum_{k \in z} f\hat{}\,(j+kN) = \frac{1}{N} \sum_{k=0}^{N-1} f(2\pi k/N)\, e^{-2\pi ijk/N} \tag{14}$$

We leave to the next section the discussion of (2), (4), (4'), and (2').

3. $L(x)$ AS A CONTOUR INTEGRAL

Consider $h \in [0, 1)$, $h \neq (1-(-)^n)/4$, $-h < x < 1-h$, and $j \geqslant 0$. We obtain, by (12), (8), and the residue theorem, with $L = L_{n,z-h}$ and $\lambda_\mu = \lambda_\mu(h)$, that

$$L(x+j) = \frac{1}{2\pi i} \int_T \frac{\Pi_{n,x+h}(z)}{\Pi_{n,h}(z)} z^{j-1}\, dz$$

$$= \left(\frac{1-h-x}{1-h}\right)^n \delta_{0,j} + \sum_{-1 < \lambda_\mu < 0} \frac{\Pi_{n,x+h}(\lambda_\mu)}{\lambda_\mu \Pi'_{n,h}(\lambda_\mu)} \lambda_\mu^j, \tag{15}$$

which yields, for $-h < x < 1-h$ and $0 \leqslant j \leqslant N-1$,

$$\begin{aligned} L^{(N)}(x+j) = {} & \left(\frac{1-h-x}{1-h}\right)^n \delta_{0,j} \\ & + \sum_{-1 < \lambda_\mu(h) < 0} \frac{\Pi_{n,x+h}(\lambda_\mu(h))\, \lambda_\mu(h)^j}{\lambda_\mu(h)\, \Pi'_{n,h}(\lambda_\mu(h))(1-\lambda_\mu(h)^N)} \\ & + \sum_{-1 < \lambda_\mu(1-h) < 0} \frac{\Pi_{n,1-h-x}(\lambda_\mu(1-h))\, \lambda_\mu(1-h)^{N-j}}{\lambda_\mu(1-h)\, \Pi'_{n,1-h}(\lambda_\mu(1-h))(1-\lambda_\mu(1-h)^N)} \end{aligned} \tag{15'}$$

by the observation that

$$L^{(N)}(x+j) = \sum_{k=0}^{\infty} L_{n,Z-h}(x+j+kN) + \sum_{k=0}^{\infty} L_{n,Z-(1-h)}(-x+N-j+kN). \tag{16}$$

We note that, in (15) and (15′),

$$\operatorname{sgn} \Pi_{n,x+h}(\lambda_\mu(h)) = (-)^{\mu+1} = \operatorname{sgn} \lambda_\mu \Pi'_{n,h}(\lambda_\mu(h)), \qquad \text{for} \quad 0 < x < 1-h,$$

and (17)

$$\operatorname{sgn} \Pi_{n,x+h}(\lambda_\mu(h)) = (-)^\mu, \qquad \text{for} \quad -h < x < 0.$$

Similar statements hold with the pair (x, h) replaced by $(-x, 1-h)$.

Formulae (2) and (4) are now immediate consequences of (15) and (17). Equations (17) also show that strict inequality holds in (4) with nonzero x.

For odd N, (15′), (17), and the fact that $\operatorname{sgn} \lambda_\mu^j = -\operatorname{sgn} \lambda_\mu^{N-j}$ visibly imply (2′), which, together with a simple calculation yields (4′), with strict inequality for nonzero x.

For even N, similar considerations with the aid of (2′) give (4′).

We mention that, for $h=0$ and n odd, the observation

$$L^{(N)}(x-j)=\sum_{k=0}^{\infty} L(x+j+kN)+\sum_{k=0}^{\infty} L(1-x+N-1-j+kN) \tag{16a}$$

gives, with $\lambda_\mu=\lambda_\mu(0)$

$$L^{(N)}(x+j)=(1-x)^n\,\delta_{0,j}+x^n\,\delta_{N-1,j} + \sum_{-1<\lambda_\mu<0} \frac{\Pi_{n,x}(\lambda_\mu)\lambda_\mu^j+\Pi_{n,1-x}(\lambda_\mu)\lambda_\mu^{N-1-j}}{\lambda_\mu \Pi'_{n,h}(\lambda_\mu)(1-\lambda_\mu^N)}, \tag{15'a}$$

which is obtained in Reimer [9].

Now we come to the result concerning the radius of convergence of (1). Formulae (1) and (6) yield, with l_2-data (f_j), say,

$$(Sf)^{(k)}(x)=\frac{1}{2\pi i}\int_{T_d}\left(\sum f_{-j}z^j\right)\frac{\partial^k}{\partial x^k}E(x,z)\frac{dz}{z}, \tag{18}$$

for $d=1$ and $k=0$, where $T_d:=\{z: |z|=d\}$ is positively oriented.

The Laurent series $F(z):=\sum f_{-j}z^j$ and the kernel function $E(x,z)$ converge in the annuli $A_{r,R}:=\{z: r<|z|<R\}$ and A_{ρ_1,ρ_2}, resp., where

$$\rho_1:=|\lambda_{m/2}|,\qquad \rho_2:=|\lambda_{(m/2)+1}|,$$

$$R:=\left(\limsup_{j\to\infty}|f_{-j}|^{1/j}\right)^{-1},\qquad r:=\limsup_{j\to\infty}|f_j|^{1/j}.$$

In general, if $R>\rho_1$ and $r<\rho_2$, i.e., if some T_d is contained in the intersection of the two annuli defined above, then (18), *with $k=0$, defines in $\mathbb{S}_n$ an interpolant to the data (f_j), as can be easily verified by* (7).

We note that $1/R\leqslant 1$ and $r\leqslant 1$ hold for data (f_j) of power growth. Equation (7) also implies that the convergence in (18), $k=0, 1,..., n-1$, is uniform in x in compact subsets of the real line. This is obtained in Reimer [10] for odd n and $h=0$.

Finally, we consider the limit of (18) as the degree n tends to infinity. For simplicity, we assume $h=0$ (n remains odd) and $k=0$. The basic fact that (see [11]) $\lim E_n(x, e^{iu})=e^{iux}$, for u in $(-\pi,\pi)$, yields then

$$\lim S_n f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum f_{-j}e^{iju}\right)e^{iux}\,du,$$

which is obtained in [12] for l_2 data f by a different method.

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